- La semana pasada, propusimos la generalización de la derivada parcial;

⤳ Derivada covariante

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda} \quad (*)$$

- Para que transforme como tensor, necesitamos:

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \quad (**)$$

- Primero, veamos cómo transforma $\nabla_\mu V^\nu$ en $(*)$:

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'}$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu}$$

$$+ \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda$$

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}$$

Transf. de la parcial

Nota: no sabemos cómo transforman $\Gamma^{\nu'}_{\mu'\lambda'}$.

· Ahora, desarrollemos el lado derecho de $(**)$:

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\nabla_{\mu}V^{\nu}=\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\partial_{\mu}V^{\nu}+\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}V^{\lambda}$$

Igualamos estos resultados:

$$\Gamma^{\nu'}_{\mu'\lambda'}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}V^{\lambda}+\frac{\partial x^{\mu}}{\partial x^{\mu'}}V^{\lambda}\frac{\partial}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\lambda}}=\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}V^{\lambda}$$

Esto se puede reducir a:

$$\boxed{\Gamma^{\nu'}_{\mu'\lambda'}=\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}-\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^{2}x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}}$$

· El segundo término arruina la transf. tensorial.

· Este término extra cancela el de la derivada parcial.

⇝ La derivada covariante es un tensor.

• ¿Qué pasa en el caso de la uno-forma?
¿Mismo $\Gamma^{\nu}_{\mu\lambda}$? <u>No necesariamente.</u>
Escribamos:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}^{\lambda}_{\mu\nu}\omega_\lambda$$

· Veamos ahora qué pasa si calculamos:

$$\nabla_\mu(\omega_\lambda V^\lambda) = (\nabla_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\nabla_\mu V^\lambda)$$

$$= (\partial_\mu \omega_\lambda) V^\lambda + \tilde{\Gamma}^{\sigma}_{\mu\lambda}\omega_\sigma V^\lambda$$

$$+ \omega_\lambda (\partial_\mu V^\lambda) + \omega_\lambda \Gamma^{\lambda}_{\mu\rho} V^\rho .$$

Pero, $\omega_\lambda V^\lambda$ es un escalar

$$\Rightarrow \nabla_\mu(\omega_\lambda V^\lambda) = \partial_\mu(\omega_\lambda V^\lambda)$$
$$= (\partial_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\partial_\mu V^\lambda)$$

Esto es cierto si

$$0 = \tilde{\Gamma}^\sigma_{\mu\lambda}\, \omega_\sigma V^\lambda + \Gamma^\sigma_{\mu\lambda}\, \omega_\sigma V^\lambda$$

$$\Rightarrow \qquad \tilde{\Gamma}^\sigma_{\mu\lambda} = -\Gamma^\sigma_{\mu\lambda} .$$

Finalmente, tenemos:

$$\boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\, \omega_\lambda}$$

Este resultado lo generalizamos:

$$\nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$$

$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots$$

$$- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_\ell} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_\ell} - \cdots$$

· En resumen,

- Derivada parcial $\longrightarrow$ tasa cambio
- Derivada covariante $\longrightarrow$ tasa cambio + corrección por curvatura.

· En Relatividad General, cada métrica define una conexión única.

· Para ver esto, notemos que la diferencia entre dos conexiones sí es un tensor:

$$S^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \underbrace{\hat{\Gamma}^{\lambda}_{\mu\nu}}$$

Si calculamos $\nabla_\mu V^\lambda - \hat{\nabla}_\mu V^\lambda$:

$$= \partial_\mu V^\lambda + \Gamma^{\lambda}_{\mu\nu} V^\nu - \partial_\mu V^\lambda - \hat{\Gamma}^{\lambda}_{\mu\nu} V^\nu$$

$$= S^{\lambda}{}_{\mu\nu} V^\nu \quad \left(\Gamma \sim \hat{\Gamma} + S, \leftarrow \text{corrección tensorial.}\right)$$

· Notemos que de igual forma pudimos definir

$$\Gamma_{\nu\mu}^{\lambda} \quad : \text{transforma igual } (\nu \to \mu).$$

⇒ Existe un tensor asociado a cualquier conexión:

$$T^{\lambda}{}_{\mu\nu} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}$$ Tensor torsión

· si la conexión es antisimétrica, T es antisim.

si Γ es simétrica $\Rightarrow T^{\lambda}{}_{\mu\nu} = 0$ "Torsion-free"

· Para que la conexión sea única, requerimos:

· $T^{\lambda}{}_{\mu\nu} = 0$

· $\nabla_{\rho} g_{\mu\nu} = 0$ (Metric compatibility)

· Esto implica:

$$\nabla_\lambda \epsilon_{\mu\nu\rho\sigma} = 0\,,$$

$$\nabla_\rho g^{\mu\nu} = 0\,.$$

Además, una derivada covariante que es "metric compatible":

$$g_{\mu\lambda} \nabla_\rho V^\lambda = \nabla_\rho (g_{\mu\lambda} V^\lambda)$$

$$= \nabla_\rho V_\mu$$

¡Si no es metric compatible, cuidado!

· Mostremos ahora, explícitamente, que existe dicha conexión (y que es única):

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0$$

· Sumando y restando obtenemos:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\,\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0$$

· Multiplicamos por $g^{\sigma\rho}$:

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$$

"Símbolos de Christoffel"

Nota: estos símbolos son cero en el espacio plano con coordenadas cartesianas, ⇝ ¡no en polares!

Para ver esto, toma:

$$ds^2 = dr^2 + r^2 d\theta^2$$

Tenemos :

$$g_{rr} = 1 \,, \quad g_{\theta\theta} = r^2$$
$$g^{rr} = 1 \,, \quad g^{\theta\theta} = r^{-2} .$$

Entonces,

$$\Gamma^{r}_{rr} = \frac{1}{2} g^{r\rho}\left(\partial_r g_{r\rho} + \partial_r g_{\rho r} - \partial_\rho g_{rr}\right)$$

Sumamos sobre $\rho = r, \theta$.

$$= \frac{1}{2} g^{rr} (\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr})$$

$$+ \frac{1}{2} g^{r\theta} (\partial_r g_{r\theta} + \partial_r g_{\theta r} - \partial_\theta g_{rr})$$

$$= \frac{1}{2}(1)(0+0-0) + \frac{1}{2}(0)(0+0-0)$$

$$\Gamma^r_{rr} = 0 .$$

Pero, $\Gamma^r_{\theta\theta} = \frac{1}{2} g^{r\rho} (\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta})$

$$= \frac{1}{2} g^{rr} (\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta})$$

$$= \frac{1}{2}(1)(0+0-2r)$$

$$= -r .$$