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2: The K-G theory and its quantization

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2} \rightsquigarrow \boxed{(\partial^2+m^2)\phi(x)=0}$$

wave-like solutions: $\phi(x) = e^{\mp ipx} = e^{\mp ip^\mu x_\mu}$

$\rightsquigarrow -p^2+m^2=0 \rightsquigarrow E^2 = |\vec{p}|^2+m^2 \doteq E_p^2 \qquad E_p = \sqrt{|\vec{p}|^2+m^2}$

General solution constructed from plane waves:

$$\boxed{\phi(x) = \int d^3\vec{p}\left(f(\vec{p})e^{-ipx} + g(\vec{p})e^{ipx}\right)}$$

K G general sol.

1.

(on-mass-shell relation $p^2=m^2$ is implied)

- Hamiltonian $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}m^2\phi^2$

- quantization: treat $\phi(\vec{x}), \pi(\vec{x})$ as operators at each spatial point; impose the canonical commutation relations

$$[\phi(\vec{x}),\phi(\vec{x}')] = 0 = [\pi(\vec{x}),\pi(\vec{x}')]$$

$$[\phi(\vec{x}),\pi(\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}')$$

these are time independent, Schrödinger operators / commutation relations; we'll see later that these translate into equal-time comm. rel. for time-dep. Heisenberg ops.

• decomposition into Fourier modes:

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{+} e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{+} e^{-i\vec{p}\cdot\vec{x}} \right)$$

Compare

note the analogy to the harm. osz.:

$$x = \frac{1}{\sqrt{2m\omega}} (a + a^+)$$

$$p = -i\sqrt{\frac{m\omega}{2}} (a - a^+)$$

here: each momentum / Fourier mode has its own pair of creation / annihilation operators $a_{\vec{p}}^{(+)}$; hence:

$$[a_{\vec{p}}, a_{\vec{p}'}] = 0 = [a_{\vec{p}}^{+}, a_{\vec{p}'}^{+}]$$

$$[a_{\vec{p}}, a_{\vec{p}'}^{+}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$$

• check consistency: rewrite

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\vec{p}} + a_{-\vec{p}}^{+}) e^{i\vec{p}\cdot\vec{x}}$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} (a_{\vec{p}} - a_{-\vec{p}}^{+}) e^{+i\vec{p}\cdot\vec{x}}$$

$$[\phi(\vec{x}), \pi(\vec{x}')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(-\frac{i}{2}\right)\sqrt{\frac{E_{p'}}{E_p}}\Big(\underbrace{[a^\dagger_{-\vec{p}}, a_{\vec{p}'}]}_{-(2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}')}$$

$$- \underbrace{[a_{\vec{p}}, a^\dagger_{-\vec{p}'}]}_{(2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}')}\Big) e^{i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{x}')}$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^3}\, i\sqrt{\frac{E_{p'}}{E_p}}\,\delta^{(3)}(\vec{p}+\vec{p}')\, e^{i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{x}')}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, i\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = i\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$[\phi_x, \pi_{x'}] = i\delta^{(3)}(x-x')$

- Hamiltonian in terms of ladder operators

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \frac{1}{2}\left(\pi^2(x) + \vec{\nabla}\phi\cdot\vec{\nabla}\phi + m^2\phi^2\right)$$

$$= \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(\vec{p}+\vec{p}')\cdot\vec{x}}\Big\{-\frac{\sqrt{E_pE_{p'}}}{4}(a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{\vec{p}'} - a^\dagger_{-\vec{p}'})$$

$$+ \frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{E_pE_{p'}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})(a_{\vec{p}'} + a^\dagger_{-\vec{p}'})\Big\}$$

<u>note</u>: $\int d^3x\, e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}')$, $\int d^3p'$ sets $\vec{p}' = -\vec{p}$, $E_{p'} = E_p$
$|\vec{p}|^2 + m^2 = E_p^2$

$a_{\vec{p}}a_{\vec{p}}$, $a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}$ terms cancel

$$= \int \frac{d^3p}{(2\pi)^3} E_p \left(\frac{1}{2} a_{\vec p} a_{\vec p}^\dagger + \frac{1}{2} a_p^\dagger a_{\vec p} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \left(a_{\vec p}^\dagger a_{\vec p} + \frac{1}{2} \underbrace{[a_{\vec p}, a_{\vec p}^\dagger]}_{\delta^{(3)}(0) \text{ infinite.}} \right)$$

$[a_{\vec p}, a_{\vec p}^\dagger] \sim$ infinite zero-point energy of infinitely many oscillators IGNORE!

$$\rightsquigarrow \boxed{H = \int \frac{d^3p}{(2\pi)^3} E_p \, a_{\vec p}^\dagger a_{\vec p}}$$

H in normal-ordered form

$$:a_{\vec p} a_{\vec p'}^\dagger: = a_{\vec p'}^\dagger a_{\vec p} \quad , \quad :a_{\vec p}^\dagger a_{\vec p'}: = a_{\vec p}^\dagger a_{p'}$$

remark:

$$\langle 0 | :a_{\vec p}^\dagger a_{\vec p'} \dots : | 0 \rangle = 0$$

- obviously: $[H, a_{\vec p'}^\dagger] = E_{p'} a_{\vec p}^\dagger$ ✓

$$[H, a_{\vec p'}] = -E_{p'} a_{\vec p'}$$ ✓

- vacuum/ground state def. by $a_{\vec p} |0\rangle = 0 \;\; \forall \vec p$

$\rightsquigarrow a_{\vec p}^\dagger$ increases the energy of a state by E_p,

$a_{\vec p}^\dagger |0\rangle$ carries energy E_p.